MATRIX TECHNIQUES

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ABSTRACT. This document collects matrix techniques for solving problems in linear algebra. None of these techniques should be applied without an understanding of why they work.

1. Elementary Invertible Matrices

The identity matrix is denoted by I.

The elementary invertible matrices are

- $E(i, j; c)$ is I except $a_{ij} = c$;
- $D(i; c)$ is I except $a_{ii} = c$;
- $P(i, j)$ is I except $a_{ii} = a_{jj} = 0$ and $a_{ij} = a_{ji} = 1$.

The inverses of the elementary invertible matrices are

- $E(i, j; c)^{-1} = E(i, j; -c)$;
- $D(i; c)^{-1} = D(i; c^{-1})$;
- $P(i, j)^{-1} = P(i, j)$.

Let E be an elementary invertible matrix. Multiplying on the left of A to form EA has the indicated effect on the rows of A. Multiplying on the right of A to form AE has the analogous effect on the columns of A.

$E(i, j; c)$	Multiply the j^{th} row by c and add to the i^{th} row
$D(i; c)$	Multiply the i^{th} row by c
$P(i, j)$	Swap the i^{th} row and the j^{th} row

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2. Gaussian Elimination

Let A denote the original matrix.
Let $B = OA$ be the result of forward elimination, where O is invertible.
Let $C = UA$ be the result of backward elimination, where U is invertible.
Let M be the modified augmented matrix obtained by solution readoff.
The basic columns of B or C are the columns containing the pivots.
The free columns of B or C are the other columns.
The basic columns of A or M correspond to the basic columns of B or C.
The free columns of A or M correspond to the free columns of B or C.
Let r be the number of basic columns of B or C.
Let k be the number of free columns of B or C.
The basic rows of O or U are the first r rows.
The free rows of O or U are the last $m - r$ rows.

Forward Elimination

(1) Start with the first nonzero column.
(2) If the top entry in the column is zero, permute with a lower row so that the top entry is nonzero (use P).
(3) Eliminate all entries below this one (use E).
(4) Repeat this process, disregarding the current top row and all rows above it.

Backward Elimination

(1) Make all pivots equal to one (use D).
(2) Starting from the right, working upward then leftward, make all entries above a pivot equal to zero (use E).

Solution Readoff

(1) insert a zero row at row i for every free variable x_i;
(2) multiply each free column by -1;
(3) add e_i to each free column;
(4) the particular solution is now the augmentation column;
(5) the homogeneous solution is now the span of the free columns.

3. Finding a Basis for Fundamental Subspaces

The four fundamental subspaces associated to A are the column space $\mathrm{col}(A)$, the row space $\mathrm{row}(A)$, the kernel $\ker(A)$, and the kernel of the transpose $\ker(A^*)$.

The primary techniques for finding a basis of these spaces are:

(F1) The basic columns of A are a basis for $\mathrm{col}(A)$.
(F2) The nonzero rows of B or C are a basis for $\mathrm{row}(A)$.
(F3) The free columns of M are a basis for $\ker(A)$.
(F4) The free rows O or U are a basis for $\ker(A^*)$.

These secondary techniques are implied by the primary technques:

(F5) The basic columns of A are a basis for $\mathrm{row}(A^*)$.
(F6) The nonzero rows of B or C are a basis for $\mathrm{col}(A^*)$.

To avoid backward elimination, row reduce A^* instead of A and apply techniques **(F2)** and **(F4)** instead of **(F1)** and **(F3)**.

4. Finding a Basis for a Span

Let $X = \{w_1, \ldots, w_n\} \subset \mathbb{R}^m$ and let $W = \mathrm{span}(X)$.
Form the $m \times n$ matrix $A = [w_1 \mid \cdots \mid w_n]$.
Reduce A and apply **(F1)**; a basis for W is a basis for $\mathrm{col}(A)$.
Reduce A^* and apply **(F2)**; a basis for W is a basis for $\mathrm{row}(A^*)$.

5. Test for Linear Independence

Let $X = \{w_1, \ldots, w_n\} \subset \mathbb{R}^m$.
If $n > m$, then X is dependent.
Form the $m \times n$ matrix $A = [w_1 \mid \cdots \mid w_n]$.
Reduce A; if $n = r$, then X is independent, otherwise it is not.

6. Test for Spanning

Let $X = \{w_1, \ldots, w_n\} \subset \mathbb{R}^m$.
If $n < m$, then X does not span $\mathbb{R}^m$.
Form the $m \times n$ matrix $A = [w_1 \mid \cdots \mid w_n]$.
Reduce A; if $m = r$, then X spans $\mathbb{R}^m$; otherwise it does not.

7. Test for a Basis

Let $X = \{w_1, \ldots, w_n\} \subset \mathbb{R}^m$.
If $n > m$, then X is not a basis.
If $n < m$, then X is not a basis.
If $n = m$, then X is a basis if and only if X spans.
If $n = m$, then X is a basis if and only if X is independent.

8. Finding the Inverse

If A is not square, it cannot be invertible.
Reduce A to B.
If $r < n$, then A is not invertible.
Reduce B to C; then $A^{-1} = U$.

9. Finding the Determinant I

If A is not square, the determinant of A is undefined.
Select any row or column and expand along it.
Along the i^{th} row:

$$\det(A) = \sum_{j=1}^{n} (-1)^{j-1} a_{ij} \det(A_{ij}).$$

Along the j^{th} column:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i-1} a_{ij} \det(A_{ij}).$$

Here, A_{ij} is the ij^{th} minor matrix of A.

10. Finding the Determinant II

If A is not square, the determinant of A is undefined.
Reduce A to B via forward elimination using E and P but not D.
If $r < n$, then $\det(A) = 0$.
If $r = n$, then B is upper triangular and $\det(B)$ is the product of the diagonal entries.
Thus $\det(A) = (-1)^p \det(B)$, where p is the number of P matrices used in forward elimination.

11. Finding Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix.
The characteristic polynomial of A is

$$\chi_A(\lambda) = \det(A - \lambda I);$$

this is a polynomial of degree n.
Then a is an eigenvalue of A if and only if a is a root of $\chi_A(\lambda)$.
To find eigenvectors associated to a, find a basis for $\ker(A - aI)$.

12. Test for Diagnolizability

Let A be an $n \times n$ matrix.
Then A is diagonalizable if and only if $\mathbb{R}^n$ has a basis of eigenvectors of A.
To diagonalize A, find a basis of eigenvectors and construct the matrix C which has these eigenvectors as columns.
Then $B = C^{-1}AC$ is diagonal.

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